Physics students' mathematical environment: Operational skills, contextual understanding, symbolic representation

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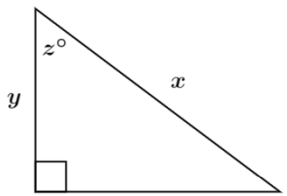
Supported in part by NSF DUE #1504986 and #1914712

Data Sources

We have given diagnostic pretests covering pre-college mathematics to over 7000 introductory physics students (non-credit; calculators allowed):

In addition, we have carried out more than 70 one-on-one problem-solving interviews with physics students to further explore the nature of students' thinking.

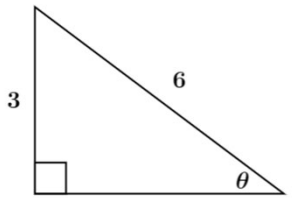
What is the length of side x?



A. $ycos(z°)$ D. $y/cos(z°)$ G. $cos(z°)/y$ J. $\sqrt{y^2+z^2}$
B. $ysin(z°)$ E. $y/sin(z°)$ H. $sin(z°)/y$ K. $\sqrt{z^2-y^2}$
C. $ytan(z°)$ F. $y/tan(z°)$ I. $tan(z°)/y$ L. y/z

(There may be more than one correct answer, but please select only ONE answer.)

What is the value of θ?



A. $cos(3/6)$ D. $cos^{-1}(3/6)$ G. $30°$ J. $27°$
B. $sin(3/6)$ E. $sin^{-1}(3/6)$ H. $45°$ K. $3/6$
C. $tan(3/6)$ F. $tan^{-1}(3/6)$ I. $60°$ L. 0.524

(There may be more than one correct answer, but please select only ONE answer.)

$cos(0°) = ?$

A. 0 B. 1 C. undefined D. 0.707 E. 0.894

(There may be more than one correct answer, but please select only ONE answer.)

$sin(90°) = ?$

A. 0 B. 1 C. undefined D. 0.707 E. 0.894

(There may be more than one correct answer, but please select only ONE answer.)

$tan(0°) = ?$

A. 0 B. 1 C. undefined D. 0.707 E. 0.894

(There may be more than one correct answer, but please select only ONE answer.)

Solve for θ.

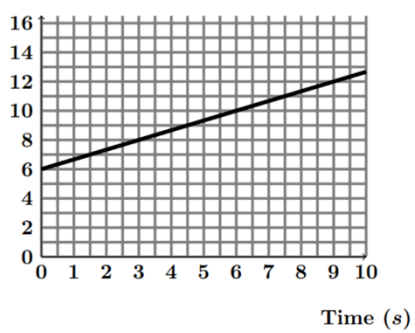
$\gamma\theta + \eta = \lambda\theta + \omega$

A. $\frac{\eta+\omega}{\gamma-\lambda}$ C. $\frac{\gamma-\lambda}{\omega-\eta}$ E. $\frac{\eta-\omega}{\gamma\lambda}$ G. $\frac{\omega-\eta}{\gamma-\lambda}$ I. $\frac{\eta-\omega+\gamma}{\lambda}$

B. $\frac{\eta-\omega}{\lambda-\gamma}$ D. $\frac{\lambda-\gamma}{\eta-\omega}$ F. $\frac{\omega-\eta}{\gamma\lambda}$ H. $\frac{\omega-\eta}{\gamma+\lambda}$ J. $\frac{\omega-\eta+\lambda}{\gamma}$

(There may be more than one correct answer, but please select only ONE answer.)

What is the slope of the graph below?



A. $\frac{1}{3}$ m/s because the object moves 1 meter in 3 seconds.

B. $\frac{1}{3}$ m/s because the line rises 1 box while it goes 3 boxes in the horizontal direction.

C. $\frac{2}{3}$ m/s because the object moves 2 meters in 3 seconds.

D. $\frac{2}{3}$ m/s because the line rises 2 boxes while it goes 3 boxes in the horizontal direction.

(There may be more than one correct answer, but please select only ONE answer.)

$\left(\frac{a}{3}\right)^3 = ?$

A. $\frac{a^3}{3}$ B. $\frac{a}{27}$ C. $\frac{a^3}{27}$

(There may be more than one correct answer, but please select only ONE answer.)

$2\left(\frac{a}{b}\right) = ?$

A. $\frac{2a}{b}$ B. $\frac{2a}{2b}$ C. $\frac{a}{2b}$

(There may be more than one correct answer, but please select only ONE answer.)

$\frac{a/b}{c^2/d} = ?$

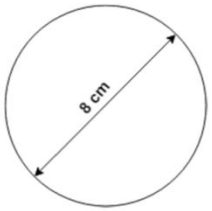
A. $\frac{ac^2}{bd}$ B. $\frac{ad}{bc^2}$ C. $\frac{bd}{ac^2}$ D. $\frac{bc^2}{ad}$

(There may be more than one correct answer, but please select only ONE answer.)

$2\left(\frac{3}{4}\right) = ?$

A. $\frac{6}{8}$ B. $\frac{12}{8}$ C. $\frac{3}{8}$ D. $\frac{3}{2}$ E. $\frac{3}{4}$

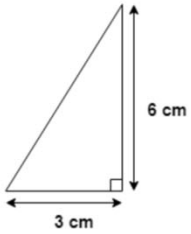
(There may be more than one correct answer, but please select only ONE answer.)



(a) Area of the circle = ?

A. 8π cm^3	F. 8π cm^2	K. 8π cm
B. 16π cm^3	G. 16π cm^2	L. 16π cm
C. 32π cm^3	H. 32π cm^2	M. 32π cm
D. 64π cm^3	I. 64π cm^2	N. 64π cm
E. 128π cm^3	J. 128π cm^2	O. 128π cm

(There may be more than one correct answer, but please select only ONE answer.)



(b) Area of the triangle = ?

A. 4.5 cm^3	F. 4.5 cm^2	K. 4.5 cm
B. 9 cm^3	G. 9 cm^2	L. 9 cm
C. 12 cm^3	H. 12 cm^2	M. 12 cm
D. 18 cm^3	I. 18 cm^2	N. 18 cm
E. 36 cm^3	J. 36 cm^2	O. 36 cm

(There may be more than one correct answer, but please select only ONE answer.)

Solve for x.

$$\frac{3}{2} = 7x$$

A. $\frac{14}{3}$ B. $\frac{3}{14}$ C. $\frac{21}{2}$ D. $\frac{21}{14}$

(There may be more than one correct answer, but please select only ONE answer.)

$$v^2 = v_0^2 + 2ad$$

$$v_0 = 0$$

$$a = \frac{\Delta v}{\Delta t}$$

$$\Delta v = 60$$

$$\Delta t = 8$$

$$v = 30$$

d =?

A. $d = 30$ B. $d = 60$ C. $d = 120$ D. $d = 240$ E. $d = 480$

(There may be more than one correct answer, but please select only ONE answer.)

$$cy = dx$$

$$a - y = bx$$

x =?

A. $\frac{ac}{d+b}$ C. $\frac{ac}{bc-d}$ E. $\frac{ac}{db}$ G. $\frac{a}{b+\frac{d}{c}}$ I. $\frac{1}{b}\left(a-\frac{d}{c}\right)$

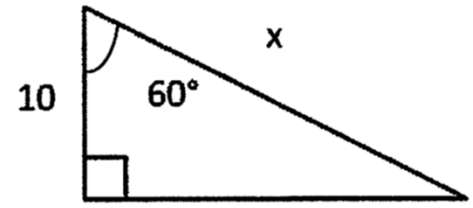
B. $\frac{ac}{d-b}$ D. $\frac{ac}{bc+d}$ F. $\frac{a}{db}$ H. $\frac{a}{b+d}$ J. $\frac{c}{d}\left(a-b\right)$

(There may be more than one correct answer, but please select only ONE answer.)

Trigonometry Questions

with samples of correct student responses

1.



What is the value of x?

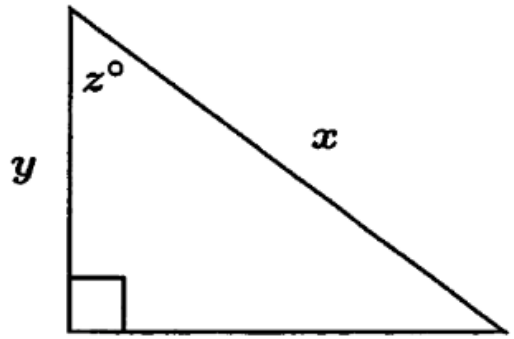
$\cos 60 = \frac{10}{x}$

$x \cos 60 = 10$

$x = \frac{10}{\cos 60}$

$= 20$

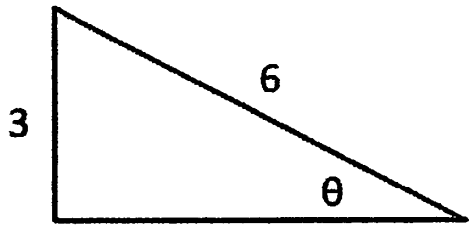
1. What is the length of side x?



$\cos z° = \frac{y}{x}$

$x = \frac{y}{\cos z°}$

3.



What is the value of θ?

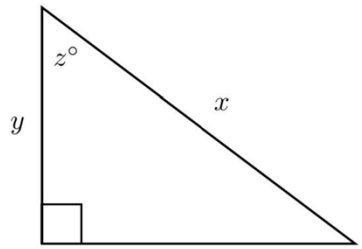
$\sin^{-1}(\theta) = \sin^{-1}\left(\frac{3}{6}\right)$

$\theta = 30^{\circ}$

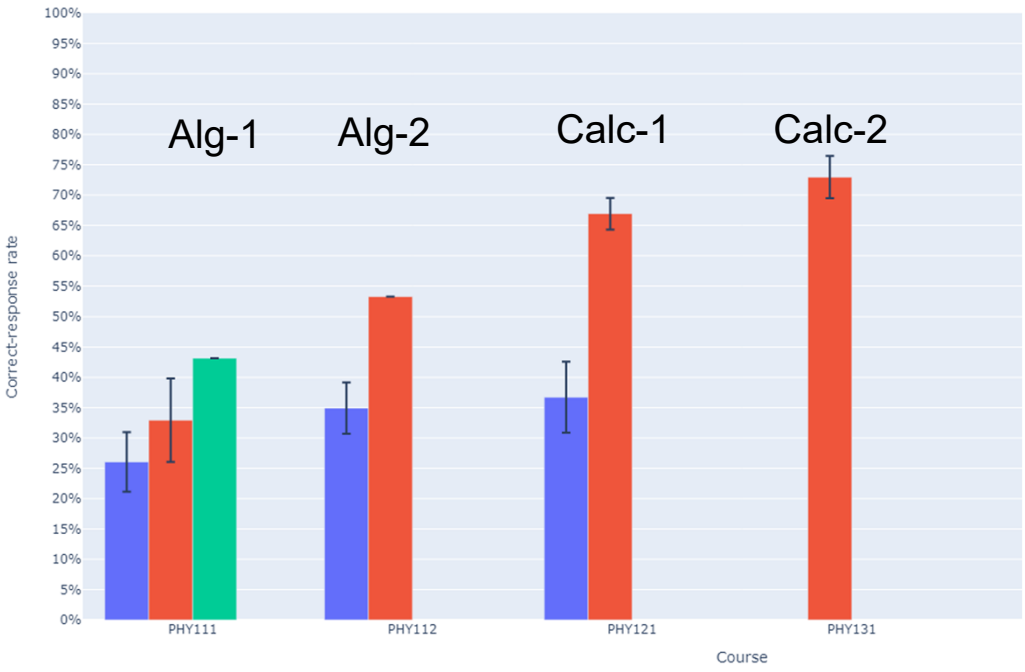
Correct-response rates (written, free-response)

(36 classes; *N* > 3000)

What is the length of side x?



Many results in the 30-60% range



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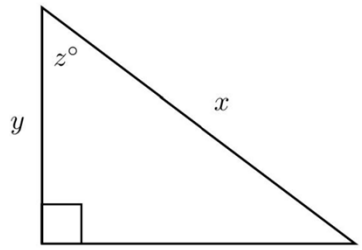

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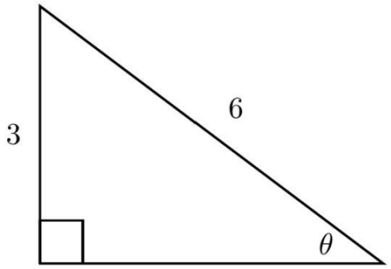
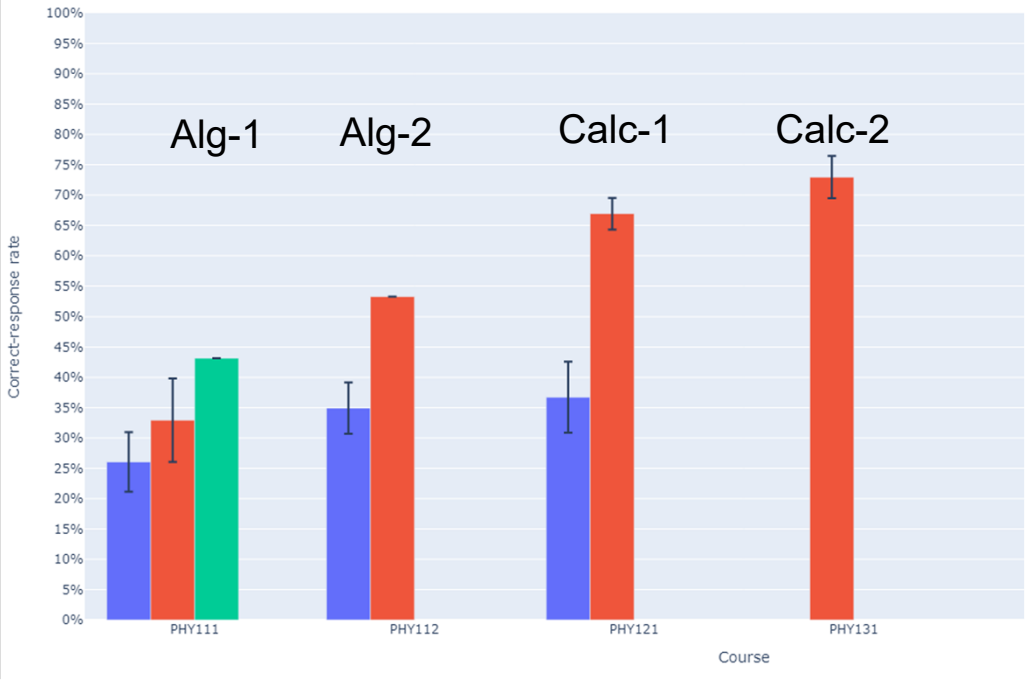
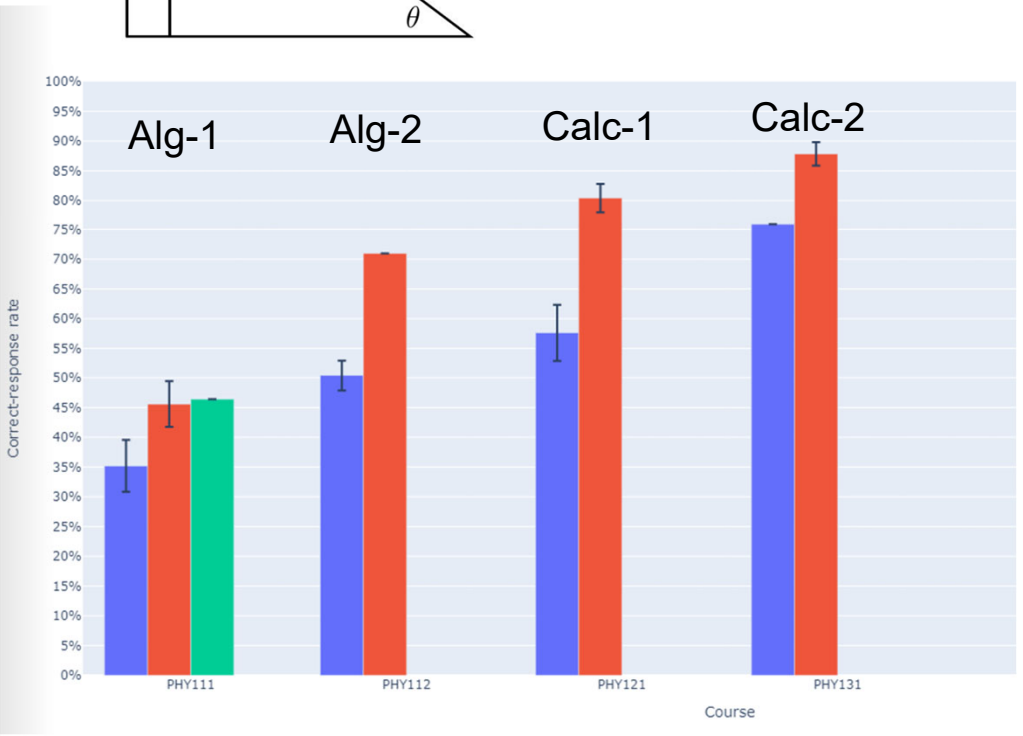


Many results in the 30-60% range





But the Problem is More Complicated…

- Weak calculational skills are only part of the problem.
- Student difficulties that *appear* to be mathematical in origin may actually be due in part to application in a *physical* context [Thompson, Manogue, Roundy, and Mountcastle, 2012; Zavala and Barniol, 2013]

“Language mismatches”

- Students are often confused by the very different symbols and techniques used in physics classes, for identical operations first seen in mathematics classes (Dray and Manogue, 1999-2004)
  - *Example:* The “area element” used in vector calculus to do area integrals looks very different in physics textbooks, compared to mathematics textbooks

$$dS = \sqrt{\left[1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]}\, dx\, dy \quad \textit{[math, general expression]}$$

$$dS = r^2 \sin\theta\, d\theta\, d\phi \quad \textit{[physics, for a sphere]}$$

Confusion can result from the nature of the symbols themselves

Solve for θ.

$$\gamma\theta + \eta = \lambda\theta + \omega$$

Solve for x.

$$ax + b = cx + d$$

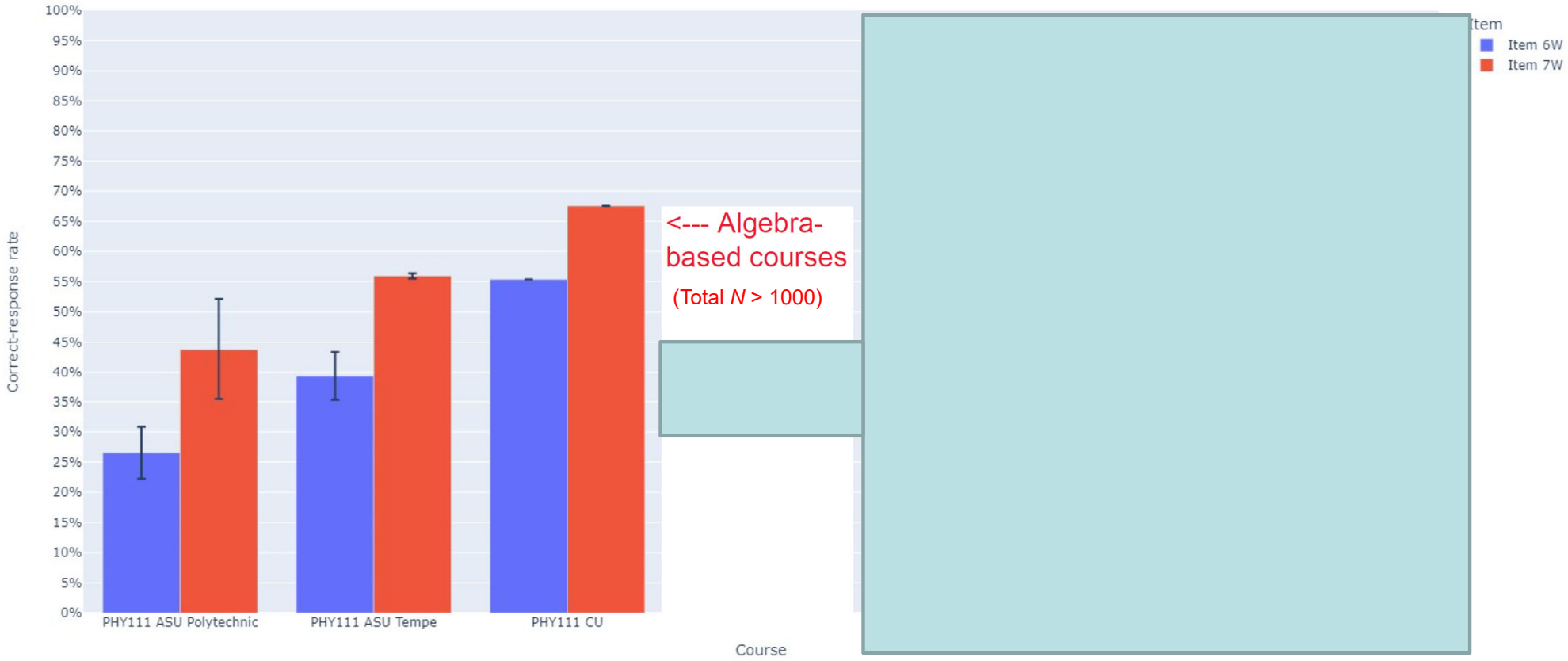
Solve for θ.

$$\gamma\theta + \eta = \lambda\theta + \omega$$

Significantly lower correct-response rates on Greek-letter version in algebra-based courses

Solve for x.

$$ax + b = cx + d$$



Solve for θ.

$$\gamma\theta + \eta = \lambda\theta + \omega$$

Significantly lower correct-response rates on Greek-letter version in algebra-based courses

Solve for x.

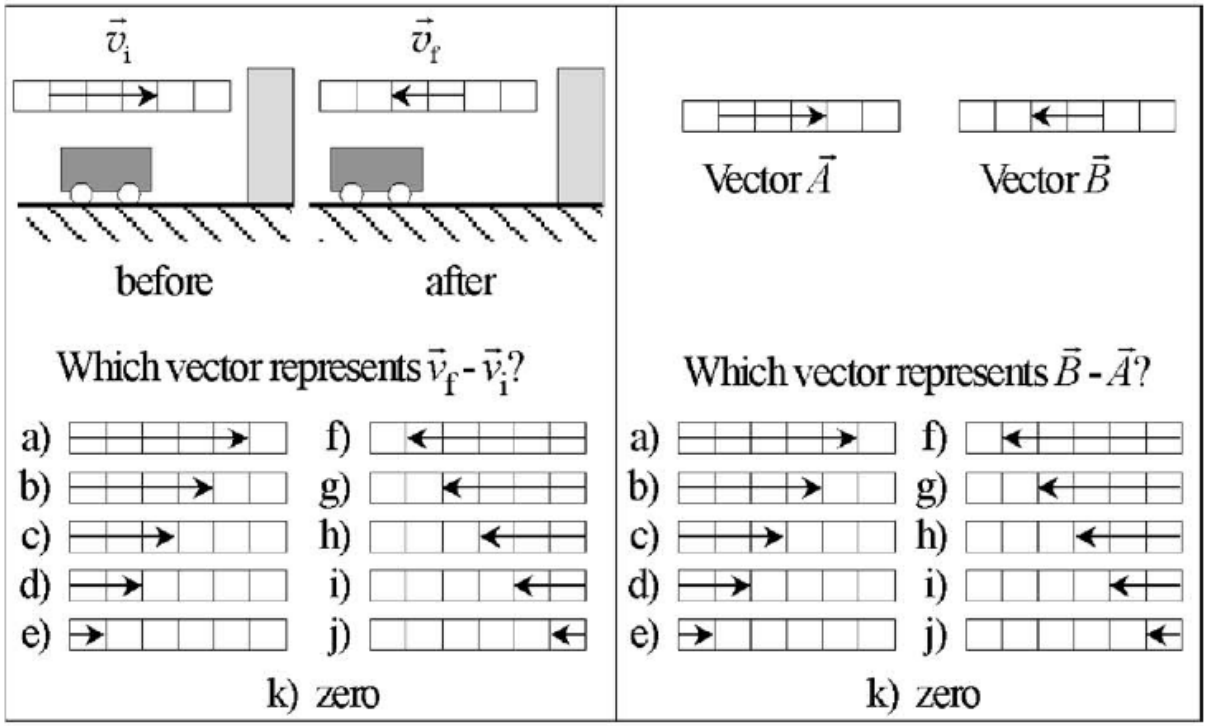
$$ax + b = cx + d$$

Physics concepts and context make a difference

- Distinguishing between a quantity (v), *change* in that quantity (Δv), and ratios of changes ($\Delta v/\Delta t$) is always challenging, but in the context of motion, additional obstacles are introduced by confusion about the *meaning* and the *distinction between* velocity and acceleration. (Trowbridge and McDermott, 1981)
- Students' difficulties in graphical interpretation in the context of motion are exacerbated by misleading intuitions drawn from objects' physical trajectories. (McDermott, Rosenquist, and Van Zee, 1987)
- Finding and comparing the "area under the curve" may be more challenging in a thermodynamics context than in a "pure math" context, e.g., in analyzing thermodynamic process represented on a pressure-volume diagram. (Christensen and Thompson, 2010-2012)

Influence of context

Shaffer and McDermott (2005) found a lower correct-response rate on a vector question in a physical context (45% correct, left diagram) than in an abstract mathematical context (65% correct, right diagram)



What factors influence ability to interpret context?

- Higher proficiency with basic calculational skills frees energy and time to apply conceptual knowledge in appropriate contexts.
- Self-checking skills can help avoid preventable errors.
- The level of reasoning skill may impact ability to interpret contexts appropriately.
- Deeper understanding of physics concepts can help guide their appropriate application in context.

Findings from >70 Interviews: Students make many “careless” errors

- During interviews, students tended to self-correct approximately 60% of their initial errors with little or no prompting, suggesting that many errors are “careless.”
- These findings suggest that increased focus on improving students’ self-checking behavior might provide significant performance dividends.
  - However, studies have shown that making these improvements is quite challenging

Relation Between Scores and Grades

- Performance on **full online diagnostic** can *approximately* predict final course grade

Low Course Grade vs. Full Diagnostic Score

Course	Campus	*N*	Overall % grade ≤ C+	Score ≥ 81% % grade ≤ C+	Score ≤ 57% % grade ≤ C+	Low-grade Ratio score ≤ 57% vs. score ≥ 81%

Low Course Grade vs. Full Diagnostic Score

Course	Campus	*N*	Overall % grade ≤ C+			
Alg-1 2021	ASU-P	78				

Alg-1: Algebra-based course, first semester

Low Course Grade vs. Full Diagnostic Score

Course	Campus	*N*	Overall % grade ≤ C+			
Alg-1 2021	ASU-P	78	26%			

Alg-1: Algebra-based course, first semester

Low Course Grade vs. Full Diagnostic Score

Course	Campus	*N*	Overall % grade ≤ C+	Score ≥ 81% % grade ≤ C+	Score ≤ 57% % grade ≤ C+	Low-grade Ratio score ≤ 57% vs. score ≥ 81%
Alg-1 2021	ASU-P	78	26%	19%	38%	2.1

Alg-1: Algebra-based course, first semester

Students who scored low on math diagnostic pretest had more "C" course grades than those who scored high

Low Course Grade vs. Full Diagnostic Score

Course	Campus	*N*				
Alg-1 2021	ASU-P	78				
Alg-1 2022	ASU-P	93				
Alg-2	ASU-P	72				
Calc-1	UWF	103				

Alg-1: Algebra-based course, first semester
Alg-2: Algebra-based course, second semester
Calc-1: Calculus-based course, first semester

ASU-P: Arizona State University, Polytechnic campus
UWF: University of West Florida

Low Course Grade vs. Full Diagnostic Score

Course	Campus	*N*	Overall % grade ≤ C+			
Alg-1 2021	ASU-P	78	26%			
Alg-1 2022	ASU-P	93	19%			
Alg-2	ASU-P	72	29%			
Calc-1	UWF	103	39%			

Alg-1: Algebra-based course, first semester
Alg-2: Algebra-based course, second semester
Calc-1: Calculus-based course, first semester

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Alg-1 2022	ASU-P	93	19%			
Alg-2	ASU-P	72	29%			
Calc-1	UWF	103	39%			

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Alg-1 2021	ASU-P	78	26%	19%	38%	2.1
Alg-1 2022	ASU-P	93	19%	8%	28%	3.4
Alg-2	ASU-P	72	29%	14%	35%	2.6
Calc-1	UWF	103	39%	26%	54%	2.1

Alg-1: Algebra-based course, first semester
Alg-2: Algebra-based course, second semester
Calc-1: Calculus-based course, first semester

ASU-P: Arizona State University, Polytechnic campus
UWF: University of West Florida

*Students who scored low on math diagnostic pretest had **consistently** more "C" course grades than those who scored high*

High Course Grade vs. Full Diagnostic Score

Course	Campus	*N*	Overall % grade ≥ A-	Score ≥ 81% % grade ≥ A-	Score ≤ 57% % grade ≥ A-	High-grade Ratio score ≥ 81% vs. score ≤ 57%
Alg-1 2021	ASU-P	78				
Alg-1 2022	ASU-P	93				
Alg-2	ASU-P	72				
Alg-2	ASU-T	129				
Calc-1	UWF	103				
Calc-2	UWF	59				

Alg-1: Algebra-based course, first semester
Alg-2: Algebra-based course, second semester
Calc-1: Calculus-based course, first semester
Calc-2: Calculus-based course, second semester

ASU-P: Arizona State University, Polytechnic campus
ASU-T: Arizona State University, Tempe campus
UWF: University of West Florida

High Course Grade vs. Full Diagnostic Score

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Alg-1 2021	ASU-P	78	35%			
Alg-1 2022	ASU-P	93	45%			
Alg-2	ASU-P	72	39%			
Alg-2	ASU-T	129	60%			
Calc-1	UWF	103	22%			
Calc-2	UWF	59	49%			

Alg-1: Algebra-based course, first semester
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Calc-1: Calculus-based course, first semester
Calc-2: Calculus-based course, second semester

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UWF: University of West Florida

High Course Grade vs. Full Diagnostic Score

Course	Campus	*N*	Overall % grade ≥ A-	Score ≥ 81% % grade ≥ A-	Score ≤ 57% % grade ≥ A-	High-grade Ratio score ≥ 81% vs. score ≤ 57%
Alg-1 2021	ASU-P	78	35%	63%	15%	4.2
Alg-1 2022	ASU-P	93	45%	67%	28%	2.4
Alg-2	ASU-P	72	39%	64%	25%	2.6
Alg-2	ASU-T	129	60%	67%	55%	1.2
Calc-1	UWF	103	22%	40%	0%	"∞"
Calc-2	UWF	59	49%	61%	38%	1.6

Alg-1: Algebra-based course, first semester
Alg-2: Algebra-based course, second semester
Calc-1: Calculus-based course, first semester
Calc-2: Calculus-based course, second semester

ASU-P: Arizona State University, Polytechnic campus
ASU-T: Arizona State University, Tempe campus
UWF: University of West Florida

*Students who scored high on math diagnostic pretest had **consistently** more "A" course grades than those who scored low*

Factors other than math preparation may influence course performance

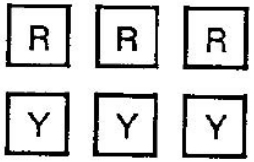
- For example:
  - Scientific reasoning skills, as measured by the Lawson Test of Scientific Reasoning
  - Physics concept knowledge, as measured by the Force Concept Inventory

Scientific reasoning skills: The 24-item Lawson test

Suppose you are given two clay balls of equal size and shape. The two clay balls also weigh the same. One ball is flattened into a pancake-shaped piece. *Which of these statements is correct?*

a. The pancake-shaped piece weighs more than the ball
b. The two pieces still weigh the same
c. The ball weighs more than the pancake-shaped piece

Six square pieces of wood are put into a cloth bag and mixed about. The six pieces are identical in size and shape, however, three pieces are red and three are yellow. Suppose someone reaches into the bag (without looking) and pulls out one piece. *What are the chances that the piece is red?*



a. 1 chance out of 6
b. 1 chance out of 3
c. 1 chance out of 2
d. 1 chance out of 1
e. cannot be determined

High Course Grade vs. Lawson Test of Scientific Reasoning Pretest Score

Course	Campus	*N*	Overall % grade ≥ A-	Top-quartile Lawson % grade ≥ A-	Bottom-quartile Lawson % grade ≥ A-	High-grade Ratio Top quartile vs. Bottom quartile
Alg-1 2021	ASU-P	73	35%	65%	17%	3.9
Alg-1 2022	ASU-P	99	45%	62%	28%	2.2
Alg-2	ASU-P	73	39%	60%	15%	4.0

Alg-1: Algebra-based course, first semester
Alg-2: Algebra-based course, second semester

ASU-P: Arizona State University, Polytechnic campus

*Students who scored high on Lawson reasoning pretest had **consistently** more "A" course grades than those who scored low*

Low Course Grade vs. Lawson Test of Scientific Reasoning Pretest Score

Course	Campus	*N*	Overall % grade ≤ C+	Top-quartile Lawson % grade ≤ C+	Bottom-quartile Lawson % grade ≤ C+	Low-grade Ratio Bottom quartile vs. Top quartile
Alg-1 2021	ASU-P	73	26%	5%	56%	11.1
Alg-1 2022	ASU-P	99	19%	10%	28%	2.9
Alg-2	ASU-P	73	29%	10%	35%	3.5

Alg-1: Algebra-based course, first semester
Alg-2: Algebra-based course, second semester

ASU-P: Arizona State University, Polytechnic campus

*Students who scored low on Lawson reasoning pretest had **consistently** more "C" course grades than those who scored high*

Physics concept knowledge: The 30-item FCI

An elevator is being lifted up an elevator shaft at a constant speed by a steel cable as shown in the figure below. All frictional effects are negligible. In this situation, forces on the elevator are such that:

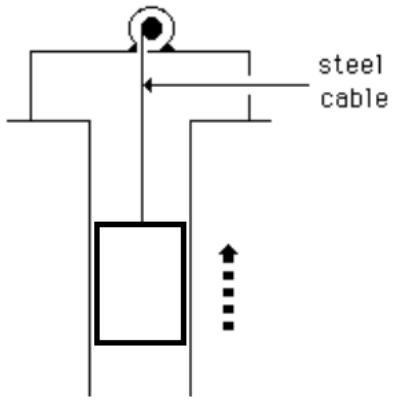
(A) the upward force by the cable is greater than the downward force of gravity.

(B) the upward force by the cable is equal to the downward force of gravity.

(C) the upward force by the cable is smaller than the downward force of gravity.

(D) the upward force by the cable is greater than the sum of the downward force of gravity and a downward force due to the air.

(E) none of the above. (The elevator goes up because the cable is being shortened, not because an upward force is exerted on the elevator by the cable).



Elevator going up
at constant speed

What Grade is Predicted by FCI Pretest Score?

- Henderson (2002), University of Minnesota (*N* > 1000)

FCI Pretest score:	**0-30%**	**63-100%**
	A: 10%	***A***: 47%
	B: 30%	***B***: 40%
	C: 46%	***C***: 9%

Students who scored high on FCI pretest had higher course grades than those who scored low

What Grade is Predicted by FCI Pretest Score?

- Meltzer (2012/13), Arizona State University (*N* > 100)

FCI Pretest score:	**0-30%**	**63-100%**
	A: 12%	***A***: 65%
	B: 44%	***B***: 22%
	C: 26%	***C***: 13%

Students who scored high on FCI pretest had higher course grades than those who scored low

Factors are correlated, but not 100%

- Outliers using one prediction method—e.g., low-scorers on the math diagnostic with high grades—can often be explained by high pretest scores on another predictor, such as the Lawson reasoning test or the FCI
- Students with low pretest scores on all diagnostics can sometimes perform well with exceptional efforts (e.g., regular class attendance, excellent class participation, consistent execution of assignments)

Summary

- Numerous factors influence students' physics course performance
- Previous preparation in calculational and reasoning skills is important, as well as physics concept knowledge
- Motivation and effort can potentially compensate for relative weaknesses in skills or knowledge preparation